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And,

$$\cos^{(\alpha)}(x) - i\sin^{(\alpha)}(x) = (-i)^{\alpha} e^{-ix}$$

Hence we obtain,

$$\cos^{(\alpha)}(x) = \frac{i^{\alpha} e^{ix} + (-i)^{\alpha} e^{-ix}}{2}$$

$$= \frac{e^{\alpha(i\pi/2)} e^{ix} + e^{\alpha(-i\pi/2)} e^{-ix}}{2}$$

Using the fact that:

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

We have:

$$\cos^{\alpha}(x) = \frac{(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2))e^{ix} + (\cos(\pi\alpha/2) - i\sin(\pi\alpha/2))e^{-ix}}{2}$$

$$= \cos(\pi\alpha/2)\frac{(e^{ix} + e^{-ix})}{2} + \sin(\pi\alpha/2)\frac{(e^{ix} - e^{-ix})}{2i}$$

$$= \cos(\pi\alpha/2)\cos x + \sin(\pi\alpha/2)\sin x = \cos\left(x + \frac{\alpha\pi}{2}\right)$$

Similar argument gives:

$$\sin^{(\alpha)}(x) = \sin\left(x + \frac{\alpha\pi}{2}\right) \qquad \blacksquare$$

Thus,

$$(-\alpha)!\,g(1) = -e\alpha\int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt + \left(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots\right)$$

Using **Lemma 3**, we get:

$$(-\alpha)!\,g(1) = (-\alpha)!\,e$$

But, $(-\alpha)!\,g(1) = h(1) = ce = (-\alpha)!e$ which implies that: $c = (-\alpha)!$.

Now, suppose that: $\alpha \le -1$. Let k be any integer smaller than $\alpha + 1$, then $\alpha - k \ge -1$ and using the equality:

$$\frac{d^{\alpha}}{dx^{\alpha}} \circ \frac{d^{\beta}}{dx^{\beta}}(f(x)) = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}(f(x)),$$

To obtain:

$$\frac{d^{\alpha}e^{x}}{dx^{\alpha}} = \frac{d^{\alpha-k}}{dx^{\alpha-k}} \circ \frac{d^{k}}{dx^{k}} e^{x} = \frac{d^{\alpha-k}}{dx^{\alpha-k}} e^{x} = e^{x}$$

Thus for any, $\alpha \in R$, $\dfrac{d^{(\alpha)}}{dx^{(\alpha)}}\left(e^{x}\right) = e^{x}$. ■

Using theorems 1 and 2, the following theorem is obtained:

Theorem 3:

For any, $\alpha \in R$, $\dfrac{d^{(\alpha)}}{dx^{(\alpha)}}\left(e^{\lambda x}\right) = \lambda^{\alpha} e^{\lambda x}$.

Corollary 1:

The fractional derivatives of order α of the functions $\sin x$ and $\cos x$ are given by

$$\sin^{(\alpha)}(x) = \sin\left(x + \frac{\alpha\pi}{2}\right), \quad \cos^{(\alpha)}(x) = \cos\left(x + \frac{\alpha\pi}{2}\right)$$

Proof:

Using the identity $\cos x + i\sin x = e^{ix}$ and according to **Theorem 3**:

$$\cos^{(\alpha)}(x) + i\sin^{(\alpha)}(x) = i^{\alpha} e^{ix}$$

Multiplying $g(x)$ by $(-\alpha)!$ we get:

$$(-\alpha)!g(x) = \cdots + \frac{(-\alpha)!x^{-2-\alpha}}{(-2-\alpha)!} + \frac{(-\alpha)!x^{-1-\alpha}}{(-1-\alpha)!} + \frac{(-\alpha)!x^{-\alpha}}{(-\alpha)!} + \frac{(-\alpha)!x^{1-\alpha}}{(1-\alpha)!} + \frac{(-\alpha)!x^{2-\alpha}}{(2-\alpha)!} + \cdots$$

and using the identity:

$$(x+n)! = (x+n)(x+n-1)\cdots(x+1)x!$$

For any x and positive integer n, the following equivalent equality is obtained:

$$(-\alpha)!g(x) = \cdots - x^{-3-\alpha}\alpha(\alpha+1)(\alpha+2) + x^{-2-\alpha}\alpha(\alpha+1) - x^{-1-\alpha}\alpha + x^{-\alpha} + \frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{2-\alpha}}{(2-\alpha)(1-\alpha)} + \frac{x^{3-\alpha}}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots$$

Now, let $h(x)$ denote the right side of the previous equality, clearly it satisfies $h'(x) = h(x)$, then $h(x) = ce^x$, where c is constant. Hence it is sufficient to prove that: $c = (-\alpha)!$. Now, let $x = 1$ then:

$$(-\alpha)!g(1) = \left(\cdots - \alpha(\alpha+1)(\alpha+2) + \alpha(\alpha+1) - \alpha\right) + \left(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots\right)$$

Two cases are considered as follows:

Case I: Let $\alpha > -1$, then the series is convergent for any $\alpha \notin \{1,2,3,\cdots\}$,

$$1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots$$

Upon using **Lemma 1**, an equality of the form is obtained:

$$-\alpha + \alpha(\alpha+1) - \alpha(\alpha+1)(\alpha+2) + \cdots = k(1) = -e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}dt$$

$$(-\alpha)! = e^{-1}\left(\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\cdots\right)+\left(-\alpha\int_0^1 e^{-1/t}t^{\alpha-1}dt\right)+e^{-1}$$

Thus,

$$\left((-\alpha)!e+\alpha e\int_0^1 e^{-1/t}\ t^{\alpha-1}\ dt\right)=1+\frac{1}{1-\alpha}+\frac{1}{(1-\alpha)(2-\alpha)}+\cdots$$

Hence,

$$(-\alpha)! = -\alpha\int_0^1 e^{-1/t}\ t^{\alpha-1}\ dt+e^{-1}\left(1+\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\cdots\right)$$ ■

Theorem 2:

For any $\alpha \in R$, $\frac{d^{(\alpha)}}{dx^{(\alpha)}}\left(e^x\right)=e^x$.

Proof:

It is sufficient to prove the theorem for $\alpha \notin Z$, because for $\alpha \in Z$, the theorem is obvious. So, assume that $\alpha \notin Z$.

Using the expansion

$$e^x=\sum_{i=-\infty}^{+\infty}\frac{x^i}{i!}=\cdots+\frac{x^{-2}}{(-2)!}+\frac{x^{-1}}{(-1)!}+\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots$$

Applying **Definition 1**,we have:

$$\frac{d^{(\alpha)}}{dx^{(\alpha)}}\left(e^x\right)=\sum_{i=-\infty}^{+\infty}\frac{x^{i-\alpha}}{(i-\alpha)!}$$

$$=\cdots+\frac{x^{-2-\alpha}}{(-2-\alpha)!}+\frac{x^{-1-\alpha}}{(-1-\alpha)!}+\frac{x^{-\alpha}}{(-\alpha)!}+\frac{x^{1-\alpha}}{(1-\alpha)!}+\frac{x^{2-\alpha}}{(2-\alpha)!}+\cdots$$

and hence the theorem can be proved if the following identity is proved:

$$\cdots+\frac{x^{-2-\alpha}}{(-2-\alpha)!}+\frac{x^{-1-\alpha}}{(-1-\alpha)!}+\frac{x^{-\alpha}}{(-\alpha)!}+\frac{x^{1-\alpha}}{(1-\alpha)!}+\frac{x^{2-\alpha}}{(2-\alpha)!}+\cdots=e^x$$

Let

$$g(x)=\cdots+\frac{x^{-2-\alpha}}{(-2-\alpha)!}+\frac{x^{-1-\alpha}}{(-1-\alpha)!}+\frac{x^{-\alpha}}{(-\alpha)!}+\frac{x^{1-\alpha}}{(1-\alpha)!}+\frac{x^{2-\alpha}}{(2-\alpha)!}+\cdots$$

Proof:
Using the definition of Gamma function, we have

$$(-\alpha)! = \Gamma(-\alpha+1) = \int_0^{+\infty} e^{-t}\, t^{-\alpha}\, dt = \int_0^{1} e^{-t}\, t^{-\alpha}\, dt + \int_1^{+\infty} e^{-t}\, t^{-\alpha}\, dt$$

For the first integral **Lemma 2** implies

$$\int_0^1 e^{-t}\, t^{-\alpha}\, dt = e^{-1}\left(\frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha(1-\alpha)} + \cdots\right) \quad (1)$$

For the second integral, using repeated integration by parts we have:

$$\int_1^{+\infty} e^{-t}\, t^{-\alpha}\, dt = e^{-1}\left(1-\alpha+\alpha(\alpha+1)-\cdots+(-1)^{n-1}\alpha(\alpha+1)\cdots(\alpha+n-1)\right)$$
$$+(-1)^n \alpha(\alpha+1)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}dt \quad (2)$$

But,

$$\int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt = \int_1^{\infty} e^{-t} t^{-\alpha-1} dt \quad (3)$$

Since $\alpha+1>0$ when $\alpha>-1$, then:

$$\int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt = \int_1^{\infty} e^{-t} t^{-(\alpha+1)} dt$$
$$= e^{-1}\left(1-(\alpha+1)+(\alpha+1)(\alpha+2)-\cdots\right.$$
$$\left.+(-1)^{n-1}(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\right)$$
$$+(-1)^n(\alpha+1)(\alpha+2)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}dt$$

Therefore,

$$\left(-\alpha\, e\int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt\right)+1 = e\int_1^{\infty} e^{-t} t^{-\alpha} dt$$

Now, from (1),(2) and (3) we have:

Proof:

Since $t^{-\alpha}e^{-t} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{n-\alpha}}{n!}$, then

(1) $$\int_0^1 e^{-t}\, t^{-\alpha}\, dt = \int_0^1\left(\sum_{n=0}^{+\infty}(-1)^n\,\frac{t^{n-\alpha}}{n!}\right)dt$$

$$= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\int_0^1 t^{n-\alpha}dt = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{1}{n-\alpha+1}$$

i.e.

(2) $$\int_0^1 e^{-t}t^{-\alpha}dt = \frac{1}{1-\alpha}-\frac{1}{1!}\frac{1}{2-\alpha}+\frac{1}{2!}\frac{1}{3-\alpha}-\frac{1}{3!}\frac{1}{4-\alpha}+\cdots$$

A proof of the following identity was reported in p.238 of Ref. [4]:

(3) $$\frac{1}{\Gamma(x+1)}+\frac{1}{\Gamma(x+2)}+\cdots = \frac{e}{\Gamma(x)}\left(\frac{1}{x}-\frac{1}{1!}\frac{1}{x+1}+\frac{1}{2!}\frac{1}{x+2}-\frac{1}{3!}\frac{1}{x+3}+\cdots\right)$$

Substituting $x = 1-\alpha$, equation (3) becomes:

(4) $$\frac{1}{\Gamma(2-\alpha)}+\frac{1}{\Gamma(3-\alpha)}+\cdots = \frac{e}{\Gamma(1-\alpha)}\left(\frac{1}{1-\alpha}-\frac{1}{1!}\frac{1}{2-\alpha}+\frac{1}{2!}\frac{1}{3-\alpha}-\frac{1}{3!}\frac{1}{4-\alpha}+\cdots\right)$$

From (2) and (4), we have:

$$\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}+\frac{\Gamma(1-\alpha)}{\Gamma(3-\alpha)}+\cdots = e\int_0^1 e^{-t}t^{-\alpha}dt$$

Thus,

$$\int_0^1 e^{-t}\, t^{-\alpha}\, dt = e^{-1}\left(\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha(1-\alpha)}+\cdots\right). \blacksquare$$

Lemma 3:

For any $\alpha > 0$,

$$(-\alpha)! = -\alpha\int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt$$
$$+\, e^{-1}\left(1+\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots\right)$$

In this paper it is proved that the fractional derivatives of order α, $\alpha \in R$, of the exponential function e^x is the exponential function e^x. This is done using the following lemmas:

Lemma 1

For any $\alpha > -1$, if

$$k(x) = -\alpha x^{1+\alpha} + \alpha(\alpha+1)x^{2+\alpha} - \alpha(\alpha+1)(\alpha+2)x^{3+\alpha} + \cdots$$

then

$$k(1) = -e\alpha \int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt\,.$$

Proof:

The function k satisfies the following differential equation

$$k'(x) = -\alpha(1+\alpha)x^{\alpha} + \alpha(\alpha+1)(\alpha+2)x^{1+\alpha} - \alpha(\alpha+1)(\alpha+2)(\alpha+3)x^{2+\alpha} + \cdots$$

$$= -\frac{k(x) + \alpha x^{1+\alpha}}{x^2} = -\frac{k(x)}{x^2} - \alpha x^{1+\alpha}$$

i.e.

$$y' + \frac{1}{x^2}y = -\alpha x^{\alpha-1}, \qquad y = k(x)$$

Also, $k(0) = 0$ because $\alpha > -1$ .Hence the solution of this differential equation is the following function:

$$k(x) = -\alpha\, e^{1/x} \int_0^x e^{-1/t}\, t^{\alpha-1}\, dt$$

So

$$k(1) = -e\alpha \int_0^1 e^{-1/t}\, t^{\alpha-1}\, dt\,. \qquad \blacksquare$$

Lemma 2:

For any $\alpha > -1$,

$$\int_0^1 e^{-t}\, t^{-\alpha}\, dt = e^{-1}\left(\frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha(1-\alpha)} + \cdots\right)$$

In [2], it is proved that, by using the definition as in Equation (1.0),

$$\frac{d^\alpha}{dx^\alpha}(c) = \frac{c}{\Gamma(1-\alpha)} x^{-\alpha}$$

$$\frac{d^\alpha}{dx^\alpha}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$$

Also, in [1] the new definition to the fractional derivatives of the constant function and the function x^n gives the same results.

Theorem 1:

If $g(x) = f(\lambda x)$, then $g^{(\alpha)}(x) = \lambda^\alpha f^{(\alpha)}(\lambda x)$

Proof:

It is sufficient to prove the theorem for $\alpha \notin Z$, because for $\alpha \in Z$, the theorem is obvious. So, assume that $\alpha \notin Z$.

Write $f(x) = \sum_{n=-\infty}^{\infty} a_n \frac{x^n}{n!}$, then

$$g(x) = f(\lambda x) = \sum_{n=-\infty}^{\infty} a_n \frac{\lambda^n x^n}{n!}$$

Then

$$g^{(\alpha)}(x) = \sum_{n=-\infty}^{\infty} a_n \frac{\lambda^n x^{n-\alpha}}{(n-\alpha)!}$$

But

$$f^{(\alpha)}(x) = \sum_{n=-\infty}^{\infty} a_n \frac{x^{n-\alpha}}{(n-\alpha)!}$$

So

$$f^{(\alpha)}(\lambda x) = \sum_{n=-\infty}^{\infty} a_n \frac{(\lambda x)^{n-\alpha}}{(n-\alpha)!} = \sum_{n=-\infty}^{\infty} a_n \frac{\lambda^{n-\alpha} x^{n-\alpha}}{(n-\alpha)!}$$

$$= \lambda^{-\alpha} \sum_{n=-\infty}^{\infty} a_n \frac{\lambda^n x^{n-\alpha}}{(n-\alpha)!} = \lambda^{-\alpha} g^{(\alpha)}(x)$$

Therefore, $g^{(\alpha)}(x) = \lambda^\alpha f^{(\alpha)}(\lambda x)$. ■

INTRODUCTION AND DEFINITIONS

A fractional derivative $\frac{d^{\alpha} f(t)}{dt^{\alpha}}$ is an extension of the familiar *n*th derivative $\frac{d^{n} f(t)}{dt^{n}}$ of the function $f(t)$. The most common definition for the fractional derivative of order $\alpha \in R$ of the function *f* is the "Riemann-Liouville integral", see [2],[3],[5]:

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{x} \frac{f(t)}{(x-t)^{\alpha+1}} dt \qquad (1.0)$$

where $\Gamma(n)$ is the Euler's Gamma Function.

For example, using equation (1.0), the (1/2)th derivatives of the functions $f(x) = x$ and $g(x) = \sqrt{x}$ can be evaluated as:

$$\frac{d^{1/2}}{dx^{1/2}} x = \frac{2\sqrt{x}}{\sqrt{\pi}} \quad \text{and} \quad \frac{d^{1/2}}{dx^{1/2}} \sqrt{x} = \frac{\sqrt{\pi}}{2}$$

In [1], a new definition of the fractional derivative of the function f of the order α, $\alpha \in R$ is defined as:

If $f(x) := \sum_{i=0}^{+\infty} a_i \frac{x^i}{i!}$ and define $(-1)! = (-2)! = \cdots = \pm\infty$, then f can be written in the form

$$f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!} \qquad (1.1)$$

The fractional derivatives of order α , $\alpha \in R$, of the function *f* is defined by:

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!} \qquad (1.2)$$

Notes:

1) For each $x > 0$, $x! = \Gamma(x+1)$.

2) For each *x*, $\frac{x^i}{i!} = 0$ for $i = -1, -2, \cdots$

3) $(-\alpha)! = \frac{\Gamma(-\alpha + m)}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)}$, $\quad m-1 < \alpha \le m$, *m* is a nonnegative integer.

3) For each $x \neq 0$, $\frac{x^{i-\alpha}}{(i-\alpha)!} \neq 0$ if α is a non-integer number.

ملخص

يهدف هذا البحث الى تقديم برهان جديد للمشتقات الكسرية من قوة $\alpha \in R$ للاقترانات الاسية $e^{\lambda x}$ والتي تساوي $\lambda^{\alpha} e^{\lambda x}$ وذلك حسب تعريف المشتقات الكسرية الذي ذكر في [1]. كذلك تم تقديم برهان للمشتقات الكسرية للاقترانات المثلثية $\cos x$ و $\sin x$ من قوة $\alpha \in R$ وهي

$\sin^{(\alpha)}(x) = \sin(x + \alpha\pi/2)$ و $\cos^{(\alpha)}(x) = \cos(x + \alpha\pi/2)$.

Abstract

In this paper a new proof of the well known fact that the fractional derivative of $e^{\lambda x}$ of order $\alpha \in R$ is equal to $\lambda^{\alpha} e^{\lambda x}$ is given according to the mentioned definition in [1]. Also, it is proved that $\sin^{(\alpha)}(x) = \sin(x + \alpha\pi/2)$ and $\cos^{(\alpha)}(x) = \cos(x + \alpha\pi/2)$.

NEW PROOFS OF FRACTIONAL DERIVATIVES OF THE EXPONENTIAL AND TRIGNOMETRIC FUNCTIONS

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